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Addition of an arbitrary number of different angular momenta

V V Mikhailov

Kazan Physical and Technical Institute of the Academy of Sciences of USSR, Kazan 420 029, USSR

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Abstract. Under the vector addition of an arbitrary number of different angular momenta, each equal to $s_1, s_2, \ldots, s_r$, the resulting angular momentum j occurs with some multiplicity. In this paper, the recurrence relations and the generating functions are obtained together with the general exact and asymptotic formulae for these multiplicities, which provide a complete solution to the problem. A comparison with the exact multiplicities is given.

1. Introduction

Quantum mechanical addition of many identical or different angular momenta often arises in various many-particle problems. Here we discuss only the combinatorial aspects of the problem—the multiplicities of the total angular momentum—and give both exact and asymptotic formulae for the multiplicities.

The addition of an arbitrary number of identical angular momenta (or, simply, spins) $s=\frac{1}{2}$ was examined from this point of view in standard books and papers, e.g. Van Vleck and Sherman (1935), Condon and Shortley (1952), Dicke (1954) and Kittel (1977). Recently, a more general case for the addition of identical arbitrary spins s was studied. It has occurred in quantum chemistry in connection with the branching diagrams (Atkins and Lambert 1976, Klein and Garcia-Bach 1977), in the analysis of multiple production of elementary particles (Pelagalli *et al* 1978) and in mathematical physics under decomposition of the direct product of irreducible representations into the sum of irreducible representations (Mikhailov 1977, Rashid 1977, Pelagalli *et al* 1978).

Here we summarise the basic results from these papers. The exact multiplicities are given by

$$Q^s_{mn}=\sum_k(-1)^k\binom{n}{k}\binom{sn-\chi k+n-m-1}{n-1}, \tag{1}$$

$$P^s_{jn}=\sum_k(-1)^k\binom{n}{k}\binom{sn-\chi k+n-j-2}{n-2}, \tag{2}$$

and the asymptotic ones

$$Q^s_{mn}=\chi^n(\pi c^s_n)^{-1/2}\exp(-m^2/c^s_n), \tag{3}$$

$$P^s_{jn}=\chi^n[(\pi c^s_n)^{1/2}(c^s_n+\tfrac{1}{2})+2c^s_n]^{-1}(2j+1)\exp(-j^2/c^s_n), \tag{4}$$

$$c^s_n=\tfrac{2}{3}ns(s+1). \tag{5}$$

Here n is the number of identical spins s, $\chi = 2s+1$, Q^s_{mn} is the number of states with a particular value of m, the z component of the total angular momentum is j, P^s_{jn} is the number of the total angular momenta j, and $\binom{a}{b}$ denotes the binomial coefficients. Here and below k represents the positive integers including zero.

In this paper we give the exact and asymptotic expressions for the multiplicities in the most general case, when the vector addition includes an arbitrary number of different spins, each occurring arbitrarily many times. We also present the recurrence relations, the generating functions and the table of exact multiplicities for comparison with the values obtained from the asymptotic formula.

2. Recurrence relations and generating functions

Let us consider the addition of n_1 spins s_1, n_2 spins $s_2, \ldots, n_r$ spins s_r. As a result the total angular momenta

$$j = j_m, j_{m-1}, \ldots, 0 \text{ (or } \tfrac{1}{2}), \qquad j_m = \sum_{l=1}^{r} n_l s_l \tag{6}$$

can occur. The number of total system states with whole magnetic number m is equal to $Q^{s_1,s_2,\ldots,s_r}_{m,n_1,n_2,\ldots,n_r}$, or for simplicity Q^s_{mn}, and the number of resulting angular momenta j is $P^{s_1,s_2,\ldots,s_r}_{j,n_1,n_2,\ldots,n_r}$, or simply P^s_{jn}.

There is a well known relation

$$P^s_{jn} = Q^s_{jn} - Q^s_{j+1,n}. \tag{7}$$

The total number of system states is equal to

$$\begin{aligned} M_n &= \chi_1^{n_1}\chi_2^{n_2}\ldots\chi_r^{n_r} \equiv \chi_1^{\omega_1 n}\chi_2^{\omega_2 n}\ldots\chi_r^{\omega_r n} \\ &= \sum_{m=-m_m}^{m_m} Q^s_{mn} = \sum_{j=0(\frac{1}{2})}^{j_m} (2j+1)P^s_{jn}, \end{aligned} \tag{8}$$

where $\chi_l = 2s_l + 1$, $m_m = j_m$, ω_l is the probability of finding spin s_l in the system and $n = \sum_l n_l$, the total number of spins.

There is an expression for Q_m (Mikhailov 1979) which can be generalised for any r

$$Q^s_{mn} = \sum_{n_{l,\mu_l}} \prod_{l=1}^{r} \frac{n_l!}{n_{l,s_l}!\, n_{l,s_l-1}! \ldots n_{l,-s_l}!}, \tag{9}$$

$$\sum_{\mu_l} n_{l\mu_l} = n_l, \qquad \sum_{l=1}^{r} \sum_{\mu_l=-s_l}^{s_l} \mu_l n_{l\mu_l} = m. \tag{10}$$

For practical calculations it is rather difficult to use (9) because of the supplementary conditions (10).

It follows from the algorithm of the angular momentum addition, that the multiplicities satisfy the recurrence relations (Mikhailov 1977, 1979, Katriel and Pauncz 1977) which may be extended to the case of arbitrary r

$$Q^s_{mn} = \sum_{\mu_l=-s_l}^{s_l} Q^s_{m-\mu_l,n'(l)}, \tag{11}$$

$$P^s_{jn} = \sum_{i=|j-s_l|}^{j+s_l} P^s_{i,n'(l)} \tag{12}$$

where

$$
\begin{aligned}
&s \equiv s_1, s_2, \ldots, s_r; \qquad n \equiv n_1, n_2, \ldots, n_r \\
&n'(l) \equiv n_1, n_2, \ldots, n_{l-1}, n_l - 1, n_{l+1}, \ldots, n_r.
\end{aligned}
\tag{13}
$$

We take $Q^s_{00} = P^s_{00} = 1$ as initial conditions.

The generating function for Q_m can be formed in two different ways:

(1) as a function of one variable x,

$$f(x) = \prod_{l=1}^{r} \left(\sum_{\mu_l} x^{\mu_l} \right)^{\omega_l n} = \sum_m Q^s_{mn} x^m, \tag{14}$$

(2) as a function of two variables x and y

$$f(x, y) = \prod_l \left(\sum_{\mu_l} x^{s_l+\mu} y^{s_l-\mu} \right)^{\omega_l n} = \sum_m Q^s_{mn} x^{sn+m} y^{sn-m}. \tag{15}$$

It is not difficult to show that the recurrence relations (11) are satisfied by the numbers Q_m from (14) or (15). The generating functions (14) and (15) for $r = 1$ have been determined accordingly by Klein and Garcia-Bach (1977) and Mikhailov (1979).

3. Exact expressions

The explicit formulae for Q_m and P_j proved to be a natural generalisation of (1) and (2) to the case of arbitrary r

$$Q^s_{mn} = \sum_k (-1)^k \binom{\boldsymbol{sn} - \boldsymbol{\chi k} + n - m - 1}{n-1} \prod_l \binom{n_l}{k_l}, \tag{16}$$

$$P^s_{jn} = \sum_k (-1)^k \binom{\boldsymbol{sn} - \boldsymbol{\chi k} + n - j - 2}{n-2} \prod_l \binom{n_l}{k_l}, \tag{17}$$

where

$$k = \sum_l k_l, \qquad n = \sum_l n_l, \qquad \boldsymbol{sn} = \sum_l s_l n_l = j_m = m_m, \qquad \boldsymbol{\chi k} = \sum_l \chi_l k_l.$$

Let all numbers n except one arbitrary number be equal to zero, then it follows that all numbers k except one are also equal to zero because of the simple property of binomial coefficients $\binom{a}{b} = 0$ if $a < b$. Therefore (16) and (17) turn into (1) and (2) yielding the desired correspondence. Moreover, when m and j are equal to their maxima $\boldsymbol{sn}$, we obtain from (16) and (17) that $Q_m = P_j = 1$ which satisfies the initial conditions. Undoubtedly similar checks cannot replace the proof for (16) and (17).

The algebraic proof for (2) has been given by Mikhailov (1977), for (1) and (2) by Katriel and Pauncz (1977). Rashid (1977) using the character theory of the group has provided another proof for (2). Now we present the new algebraic proof for the general formulae (16) and (17).

It is founded on the fact that the multiplicities Q_m must satisfy r recurrence relations (11). With the help of a simple change of indices, without loss of generality, we put $l = 1$ in (11) and consider the quantity

$$D = \sum_{\mu_1} Q^s_{m+\mu_1, n} = \sum_{\mu_1} \sum_k (-1)^k \binom{\boldsymbol{sn} - \boldsymbol{\chi k} + n - m - \mu_1 - 1}{n-1} \prod_l \binom{n_l}{k_l}.$$

Introducing for conciseness

$$a = \boldsymbol{sn} - (\boldsymbol{\chi k})' + n - m,$$

$$b = \boldsymbol{sn} - \boldsymbol{\chi k} + n - m - 1 = a - \chi_1 k_1 - 1,$$

$$\boldsymbol{\chi k} = (\boldsymbol{\chi k})' + \chi_1 k_1,$$

we have

$$D = \sum_{k} (-1)^k \left[\prod_{l} \binom{n_l}{k_l}\right] \sum_{\mu_1} \binom{b-\mu_1}{n-1}$$

$$= \sum_{k} (-1)^k \left[\prod_{l} \binom{n_l}{k_l}\right] \left[\binom{b+s_1+1}{n} - \binom{b-s_1}{n}\right].$$

Here we used the property of binomial coefficients

$$\sum_{\mu=-s}^{s} \binom{b+\mu}{n-1} = \binom{b+s+1}{n-1} - \binom{b-s}{n}. \tag{18}$$

Continuing the transformation of D we divide the terms of summation over k into parts so that each part includes terms with the same indices $k_2, k_3, \ldots, k_r$ and with all possible k_1. The operator

$$C = \sum_{k_2, k_3, \ldots, k_r} (-1)^{k_2+k_3+\ldots+k_r} \binom{n_2}{k_2}\binom{n_3}{k_3} \cdots \binom{n_r}{k_r},$$

is introduced for the sake of brevity and after that D is transformed into

$$D = C \sum_{k_1} (-1)^{k_1} \binom{n_1}{k_1} \left[\binom{a-\chi_1 k_1 + s_1}{n} - \binom{a - \chi_1 k_1 - s_1 - 1}{n}\right]$$

$$= C\left\{\binom{n_1}{0}\left[\binom{a+s_1}{n} - \binom{a-s_1-1}{n}\right] - \binom{n_1}{1}\left[\binom{a+s_1-(2s_1+1)}{n}\right.\right.$$

$$\left. - \binom{a-s_1-1-(2s_1+1)}{n}\right] + \binom{n_1}{2}\left[\binom{a+s_1-2(2s_1+1)}{n}\right.$$

$$\left.\left. - \binom{a-s_1-1-2(2s_1+1)}{n}\right] - \ldots\right\}.$$

Using (18) for $s = 0$

$$\binom{n}{i} + \binom{n}{i+1} = \binom{n+1}{i+1} \tag{19}$$

we arrive at

$$D = C\left\{\binom{n_1+1}{0}\binom{a+s_1}{n} - \binom{n_1+1}{1}\binom{a-s_1-1}{n} + \binom{n_1+1}{2}\binom{a-3s_1-2}{n} - \ldots\right\}$$

$$= C \sum_{k_1} (-1)^{k_1} \binom{n_1+1}{k_1}\binom{a+s_1-\chi_1 k_1}{n}$$

$$= \sum_{k} (-1)^k \binom{n_1+1}{k_1}\binom{n_2}{k_2}\binom{n_3}{k_3} \cdots \binom{n_r}{k_r}\binom{\boldsymbol{sn} + s_1 - \boldsymbol{\chi k} + (n+1) - m - 1}{(n+1)-1}$$

$$= Q^{s}_{m, \boldsymbol{n}''(1)} \equiv Q^{s}_{m, n_1+1, n_2, n_3, \ldots, n_r}.$$

Thus we have proved that the numbers Q_m from (16) satisfy the recurrence relations (11); we also take into account the fact that the multiplicity's initial condition $Q^s_{00} = 1$ automatically follows from (17). Therefore we have completed the proof of (16). After that the correctness of (17) is easily established with the help of basic property (7) and equation (19). The additional check-up of formulae (16) and (17) has been made by means of a comparison with the exact multiplicities for $r = 2, 3, 4$ and $n = 1, 2, \ldots, 10$. However, we have not included the tables here.

4. Some properties of multiplicities

4.1.

Multiplicities Q_m defined in (16) must satisfy relation

$$Q_{-m} = Q_m \tag{20}$$

which can easily be checked by explicit calculations. The related condition for P_j from (17)

$$P_{-j-1} = -P_j \tag{21}$$

is less evident, but it is not difficult to prove equation (21) using (7) and (20):

$$P_{-j-1} = Q_{-j-1} - Q_{-j} = -(Q_j - Q_{j+1}) = -P_j.$$

Exact formulae (16) and (17) also contain other natural restrictions on multiplicities

$$\begin{array}{lll} Q_m = 0 & \text{when } m > m_m & \text{or } m < -m_m, \\ P_j = 0 & \text{when } j > j_m & \text{or } j < -j_m - 1. \end{array} \tag{22}$$

One may connect equation (21) with an angular momentum relation coming into existence under the mirror reflection of the coordinate system. In particular, it was stated by Jucys and Bandzaitis (1965) that 'the matrix element of the square of the angular momentum operator does not change under the substitution $j \to -j-1$', which corresponds to the coordinate system reflection in the xy plane.

4.2.

We can generalise the numbers P^s_{jn} introduced by Mikhailov (1977) to the case of arbitrary r

$$P^{s\nu}_{jn} = \sum_k (-1)^k \binom{sn - \boldsymbol{\chi k} + n - j - \nu - 2}{n - \nu - 2} \prod_l \binom{n_l}{k_l}. \tag{23}$$

The basic property of these numbers

$$P^{s\nu}_{jn} = P^{s,\nu-1}_{j,n} - P^{s,\nu-1}_{j+1,n} \tag{24}$$

generalises condition (7). We can prove equation (24). Let $c = \boldsymbol{sn} + n - \boldsymbol{\chi k} - j - \nu$, $d = n - \nu$, then with the help of (19) we have

$$\begin{aligned} P^{s,\nu-1}_{jn} - P^{s,\nu-1}_{j+1,n} &= \sum_k (-1)^k \left[\binom{c-1}{d-1} - \binom{c-2}{d-1}\right] \prod_l \binom{n_l}{k_l} \\ &= \sum_k (-1)^k \binom{c-2}{d-2} \prod_l \binom{n_l}{k_l} = P^{s\nu}_{jn}. \end{aligned}$$

The multiplicities Q_m and P_j defined in the previous paragraph represent particular cases of these general numbers $P_{jn}^{s\nu}$

$$P_{jn}^{s} = P_{jn}^{s,0}, \qquad Q_{mn}^{s} = P_{m,n}^{s,-1}. \tag{25}$$

4.3.

Following Mikhailov (1977) we can now write some new formulae of summation which may be useful in statistical investigation of many-spin systems. Having introduced $\boldsymbol{sn} - \boldsymbol{\chi k} = q$, $n-2=t$ we can put condition (8) in the form

$$\sum_j (2j+1) \sum_k (-1)^k \binom{q+t-j}{t} \prod_l \binom{n_l}{k_l} = \prod_l \chi_l^{n_l}. \tag{26}$$

If we pay attention to the relations

$$\sum_j (2j+1)\binom{q+t-j}{t} = \begin{cases} \dfrac{2q+t+2}{t+2}\dbinom{q+t+1}{t+1}, & j=0,1,\ldots \\[2ex] \dfrac{2q+2t+3}{t+2}\dbinom{q+t+\frac{1}{2}}{t+1}, & j=\frac{1}{2},\frac{3}{2},\ldots \end{cases} \tag{27}$$

we can take the sum over j in (26). As a result we obtain

$$\sum_k (-1)^k \frac{2\boldsymbol{sn} - 2\boldsymbol{\chi k} + n}{n} \binom{\boldsymbol{sn} - \boldsymbol{\chi k} + n - 1}{n-1} \prod_l \binom{n_l}{k_l} = \prod_l \chi_l^{n_l} \tag{28}$$

where $\boldsymbol{sn}$ is integer

$$2\sum_k (-1)^k \binom{\boldsymbol{sn} - \boldsymbol{\chi k} + n - \frac{1}{2}}{n} \prod_l \binom{n_l}{k_l} = \prod_l \chi_l^{n_l} \tag{29}$$

where $\boldsymbol{sn}$ is half-integer. Using the resembling method it is possible to take other sums

$$\sum_j P_{jn}^{s} = \sum_k (-1)^k \binom{\boldsymbol{sn} - \boldsymbol{\chi k} + n - 1}{n-1} \prod_l \binom{n_l}{k_l} \tag{30}$$

where $\boldsymbol{sn}$ is integer,

$$\sum_j P_{jn}^{s} = \sum_k (-1)^k \binom{\boldsymbol{sn} - \boldsymbol{\chi k} + n - \frac{3}{2}}{n-1} \prod_l \binom{n_l}{k_l} \tag{31}$$

where $\boldsymbol{sn}$ is half-integer.

5. Asymptotic expressions

The asymptotic formula for one kind of spin ($r=1$) has been obtained by Klein and Garcia-Bach (1977) and by Mikhailov (1979). Our derivation of the asymptotic formulae for arbitrary r will be analogous to the method which has been used by Klein and Garcia-Bach for $r=1$.

Using (14) we have after the substitution $x = \exp(\mathrm{i}\theta)$ another form of the generating function

$$f(\theta) = \prod_l \left[\sum_{\mu_l} \exp(\mathrm{i}\mu_l\theta)\right]^{\omega_l n} = \sum_m Q_{mn}^{s} \exp(\mathrm{i}m\theta). \tag{32}$$

Using the orthogonality of the functions $\exp(im\theta)$ under integration over the segment $(-\pi, \pi)$ we have

$$Q^s_{mn} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\prod_l\left[\sum_{\mu_l}\exp(\mathrm{i}\mu_l\theta)\right]^{\omega_l n}\exp(-\mathrm{i}m\theta)\,\mathrm{d}\theta. \tag{33}$$

Comparing the two functions

$$A(\theta) = \sum_{\mu=-s}^{s}\exp(\mathrm{i}\mu\theta) = \sum_{\mu}[1-\mu^2\theta^2/2!+\mu^4\theta^4/4!-\mu^6\theta^6/6!+\ldots]$$
$$= (2s+1)[1-s(s+1)\theta^2/6+s(s+1)(3s^2+3s-1)\theta^4/360$$
$$-s(s+1)(3s^4+6s^3-3s+1)\theta^6/15120+\ldots],$$
$$B(\theta) = (2s+1)\exp[-s(s+1)\theta^2/6]$$
$$= (2s+1)[1-s(s+1)\theta^2/6+s^2(s+1)^2\theta^4/72-s^3(s+1)^3\theta^6/1296+\ldots],$$

we can see that (i) the maximum values of these functions are $A(0)=B(0)=2s+1$, (ii) $A(\theta)\approx B(\theta)>1$ for $\theta\leqslant 1$, (iii) $|A(\theta>1)|\ll A(\theta\leqslant 1)$ and $B(\theta>1)\ll B(\theta\leqslant 1)$. For these reasons, the two integrals

$$\int_{-\pi}^{\pi}A^n(\theta)\,\mathrm{d}\theta, \qquad \int_{-\pi}^{\pi}B^n(\theta)\,\mathrm{d}\theta$$

are getting more closely approximate for increasing values of n. Therefore it is possible to make the substitution in (33):

$$\prod_l\left[\sum_{\mu_l}\exp(\mathrm{i}\mu_l\theta)\right]^{\omega_l n} \approx \exp(-n\zeta\theta^2/6)\prod_l \chi_l^{\omega_l n},$$
$$\zeta = \langle s^2\rangle = \sum_l \omega_l s_l(s_l+1). \tag{34}$$

With the help of standard formula (Gradshteyn and Ryzhik 1962) we can carry out integration in (34)

$$Q^s_{mn} = \left(\prod_l \chi_l^{\omega_l n}\right)(3/2\pi n\zeta)^{1/2}\exp(-3m^2/2n\zeta). \tag{35}$$

Table 1. Sequences of parameters extracted from exact multiplicities Q^s_{mn} for $r=2$: $s_1=\frac{1}{2}$, $s_2=1$; $\omega_1=\omega_2=\frac{1}{2}$. For comparison we give values of the parameter from the asymptotic expression (34): $\zeta=\frac{11}{8}=1.375$.

n	4	6	8	10	12	14	16	18	20	22	24
ζ_n	1.68	1.58	1.54	1.50	1.48	1.465	1.457	1.445	1.440	1.433	1.429

Table 2. Sequence of parameters for $r=2$: $s_1=\frac{1}{2}$, $s_2=1$; $\omega_1=\frac{1}{3}$, $\omega_2=\frac{2}{3}$. For comparison we give the values of the parameter from the asymptotic expression (34): $\zeta=\frac{19}{12}=1.583$.

n	3	6	9	12	15	18	21
ζ_n	1.95	1.81	1.73	1.70	1.67	1.66	1.64

Table 3. Comparison of exact (A) and approximate (B) multiplicities Q^{s}_{mn}, from equation (35), for the case $r=3$: $s_1=\frac{1}{2}$, $s_2=1$, $s_3=\frac{3}{2}$; $\omega_1=\frac{1}{2}$, $\omega_2=\omega_3=\frac{1}{4}$. C is the relative percentage deviation of exact and approximate multiplicities.

m	A	B	C	A	B	C	A	B	C	A	B	C
				289	338	17				2 895 558	2 852 101	−1.5
7	1	3	160				54 560	54 016	−1.0			
				873	889	1.8				5 226 470	5 090 450	−2.6
6	8	10	26				109 254	105 815	−3.1			
				2 112	2 034	−3.6				8 583 435	8 363 845	−2.5
5	32	31	−1.6				193 288	186 917	−3.2			
				4 247	4 054	−4.5				12 890 710	12 650 680	−1.8•
4	86	79	−7.2				304 744	297 728	−2.3			
				7 269	7 040	−3.1				17 771 940	17 614 910	−0.9
3	174	164	−5.2				431 576	427 626	−0.9			
				10 755	10 648	−1.0				22 557 661	22 579 030	0.1
2	280	276	−1.2				551 549	553 837	0.4			
				13 895	14 030	0.9				26 415 160	26 643 330	0.8
1	369	377	2.1				638 192	646 804	1.3			
				15 771	16 105	2.1				28 575 320	28 942 100	1.3
0	404	418	3.4				669 860	681 140	1.6			
$n\rightarrow$	4+2+2=8			6+3+3=12			8+4+4=16			10+5+5=20		

This is the asymptotic expression for multiplicities Q_m in the general case of arbitrary r, n, s_l, ω_l. The expression for angular momentum multiplicities P_j turned out, corresponding to the normal distribution (35), to be

$$P^s_{jn} = P^s_{0n}(2j+1)\exp(-j^2/c), \tag{36}$$

$$P^s_{0n} = \left(\prod_l \chi_l^{\omega_l n}\right)[\sqrt{\pi c}(c+\tfrac{1}{2})+2c]^{-1}, \tag{37}$$

$$c = \tfrac{2}{3}n\langle s^2\rangle. \tag{38}$$

Since n is large and frequently $\langle s^2\rangle > 1$ the number $\frac{1}{2}$ in (37) may be omitted.

We have used several approximations to derive (35). Therefore it is desirable to compare (35) with the exact multiplicities calculated by other methods; that is, recurrence relations (11) or the exact expression (16). The comparison of (34) with the exact values is given in tables 1 and 2. We use the following method of comparison. Let us consider the addition of two kinds of spins $s_1 = \frac{1}{2}$ and $s_2 = 1$ with equal (table 1) and different (table 2) probabilities to encounter a corresponding spin. Provided $n = n_1 + n_2$ increase, the exact multiplicities Q^s_{0n} and Q^s_{1n} for integer $\boldsymbol{sn}$ (or $Q^s_{\frac{1}{2},n}$ and $Q^s_{\frac{3}{2},n}$ for half-integer $\boldsymbol{sn}$) are calculated. Using these pairs of multiplicities we can determine with the help of (35) the parameters ζ_n. It is the sequences of ζ_n which are represented in tables 1 and 2. Both sequences apparently tend to definite limits and we may assert that these limits are equivalent to ζ from (34). In table 3 the exact multiplicities Q^s_{mn} and the approximate ones predicted by equations (34) and (35) are written together with the relative percentage deviations ΔQ^s_{mn}. It must be noted that the relative deviations ΔQ^s_{mn} asymptotically tend to zero for increasing n, and are small for $m \leqslant \boldsymbol{sn}/2 = m_m/2$.

6. Conclusions

The basic results of this paper are represented in equations (16), (17), (35) and (36). In these expressions the problem of the exact and asymptotical enumeration of angular momentum states arising under the addition of spins of different kinds are solved completely. These results entirely agree with the more particular cases previously published by different authors.

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